ABOUT PENDULUM P.L. KAPITSA OUTSIDE
AND IN THE AREA OF PARAMETRIC RESONANCE

V. G. Shironosov

Based on the resonant theory of dynamical Poincare systems, the motion is studied P.L. Kapitsa's inverted pendulum with a vibrating suspension point outside and in the zone of parametric resonance. General formulas for finding periodic motions and studying their stability without assuming smallness are obtained in an analytical form. Amplitudes of the oscillating pendulum.

Using the example of two types of solutions (stable (2: 1) and unstable (1: 1)), we obtained in the analytical form, the conditions for the occurrence of chaos and bifurcation points 2:1 <> 1:1 for inverted pendulum.

The importance of studying the dynamic stability of unstable states is noted nonlinear systems like a pendulum outside and in the zone of linear parametric resonance for holding and trapping atomic particles in electrodynamic traps.

The problem of the dynamics of a pendulum with a vibrating suspension point for a long time attracts attention [1-13]. This is due to the fact that the corresponding equation as a model

\[ \ddot{x} + \varepsilon \dot{x} + (\varepsilon_0 + \varepsilon_1 \cos \tau) \sin x - \varepsilon_2 \cos (\tau + \varphi) \cos x = 0, \quad (1) \]

it is quite often found in various fields of physics: mechanics, electrodynamics, plasma physics, etc. In particular, for \( \varepsilon = 0 \), where \( a_0 \) is the acceleration of gravity, \( a_{1(\pm)} \) is the amplitude of the longitudinal (transverse) vibration, \( I \) is the length of the pendulum, for a particle with an intrinsic magnetic moment \( \mu_0, \varepsilon_0 = \mu_0 H_0/I \), where \( T \) is the moment of inertia, \( H_0 \) is the intensity of the constant magnetic field, and \( H_{1(\pm)} \) is the amplitude of the variable of the longitudinal (transverse) pumping magnetic field, \( \varphi = \text{const}, \tau = \omega t \). For small deflection angles \( x \) and \( e_{-1} = 0 \), equation (1) reduces to the well-known hurray to Mathieu's theory, which admits a stable state of an inverted pendulum \((e_0 < 0, e_{1(\pm)} = 0)\) outside the zone of parametric resonance. In 1950, P. L. Kapitsa [2], using the approximate solution method, described and experimentally demonstrated this effect. Based
on numerical modeling, the authors of \cite{12} found stable parametrically excited oscillations of an inverted pendulum in the resonance zone. Later, \cite{1,7}, the corresponding dependences of the oscillation amplitudes on $s_0$, $e_1$ were obtained.

In addition to the above, many other non-trivial solutions were considered: vibrational, vibrational-rotational \cite{1,7,11,12}; the emergence of chaos \cite{8,10}, etc. The search for solutions (1), as a rule, for various cases was carried out using various methods (Cesari \cite{4,6}, Krylov-Bogolyubov \cite{11}, through action-angle variables \cite{8}, etc. \cite{14,15} with the expansion of $\sin x$, $\cos x$ in a series in powers of smallness $x$. Such a variety of methods made it difficult to stitch together particular solutions, interpret the obtained results, and understand the causes of chaos and bifurcations in systems described by equations of type (1).

Therefore, given the two provisions of Poincare \cite{13,75} that "... periodic solutions are the only breach through which we could try to penetrate into an area that was considered inaccessible "(I) and that" ... The periodic solution can disappear only by merging with another periodic solution ", that is," ... Periodic solutions disappear in pairs like the real roots of algebraic equations "(II), we use a generalization of the corresponding methods to find and study the stability of periodic solutions (1) from critical points of the action function \cite{13,16-23}.

To do this, we rewrite equation (1) in Lagrangian form

$$
\frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} + \frac{\partial F}{\partial x^2} = 0,
$$

$L = T - U$, $T = \dot{x}_0^2/2$, $F = \varepsilon_1 \dot{x}_0^2/2$,

$$
U = -(e_0 + e_1 \cos \tau) \cos x - e_{-1} \cos (\tau + \varphi) \sin x.
$$

In the general case, $x$ can be a vector and $U = U (\bar{x}, \tau)$. We will seek a solution to (2) near a periodic solution at a frequency in the form of a series

$$
x = x_0 + \sum_{n=1}^{\infty} \left[ x_n \cos (n \omega \tau) + \frac{y_n}{n \omega} \sin (n \omega \tau) \right],
$$

where $x_0$, $x_n$, $y_n$ in the general case $f(x)$.

Given the dependence $x$, $x = f(x_k, y_k, x_k, y_k)$, we can obtain the following shortened equations in the approximation of slowly varying amplitudes $x_k$, $y_k$ for the period $2\pi/\alpha$:

$$
\dot{x}_k \approx \frac{\partial S}{\partial y_k} \frac{\partial R}{\partial x_k},
$$

$$
y_0 = \dot{x}_k, \quad k = 1, 2, \ldots, \infty
$$

$$
S = s - y_k^2, \quad s = \langle L \rangle = \frac{\pi R}{2\gamma} \int_0^{2\pi} L d\tau,
$$

$$
R = \frac{\omega_k}{2} \left[ y_1^2 + \frac{1}{2} \sum_{\ell=1}^{\infty} (\varepsilon_1 + \varphi) \right].
$$
When deriving (6), the formulas were taken into account and conditions for the extremality of the action function (2). In the amplitude–phase variables, equations (6) take the form

\[
\begin{align*}
\frac{\delta L}{\delta x} & \equiv \left\langle \frac{\partial L}{\partial \psi} \cos \left( \omega t \right) + \frac{\partial L}{\partial x} \cos \left( \omega t \right) \rightangle, \\
\frac{\delta L}{\delta y} & \equiv \left\langle \frac{\partial L}{\partial \psi} \sin \left( \omega t \right) + \frac{\partial L}{\partial x} \sin \left( \omega t \right) \rightangle, \\
\frac{\delta L}{\delta \psi} & \equiv \left\langle \frac{\partial L}{\partial \psi} \rightangle, \\
\frac{\delta L}{\delta x} & \equiv \left\langle \frac{\partial L}{\partial x} \rightangle.
\end{align*}
\]

\[
\alpha \equiv \alpha - \omega \left( 2\dot{\alpha} + \dot{\psi} \right) \sin \left( \omega t \right) + \left( 2\dot{\psi} - \omega^2 x \right) \cos \left( \omega t \right)
\]

and conditions for the extremality of the action function (2). In the amplitude–phase variables, equations (6) take the form

\[
\begin{align*}
\dot{r}_a & \equiv \frac{1}{\omega_a} \frac{\partial S}{\partial x_a}, \quad \dot{r}_b \equiv - \frac{1}{\omega_b} \frac{\partial S}{\partial y_b} - 2z r_a, \\
x_a & = r_a \cos \psi, \quad y_a \sin \psi = r_a \sin \psi, \\
x & = x_0 + \sum_{n=1}^{\infty} r_n \cos \left( \omega t - \psi_n \right).
\end{align*}
\]

In action-angle variables

\[
\begin{align*}
\dot{\psi}_a & \equiv \frac{\partial S}{\partial x_a}, \quad \dot{\chi} \equiv - \frac{\partial S}{\partial y_a} - 2z \chi, \\
x & = x_0 + \sum_{n=1}^{\infty} \left( \frac{2z}{n} \right)^{1/2} \cos \left( \omega t - \psi_n \right).
\end{align*}
\]

It is easy to show that, to a first approximation, the Krylov–Bogolyubov method I\textsuperscript{11,14} and the S-function method for \( \eta = 1 \) lead to the same shortened equations for \( r_1 \) and \( \phi_1 \). To do this, it is sufficient to substitute (6) into (15) and take into account the equalities \( \left\langle \partial U / \partial r_1 \right\rangle \cong \left\langle \partial U / \partial x \cos \left( \alpha \tau - \phi_1 \right) \right\rangle, \quad \left\langle \partial U / \partial \phi_1 \right\rangle \cong \left\langle \partial U / \partial x \sin \left( \alpha \tau - \phi_1 \right) \right\rangle \). The parameter of smallness in both cases will be the relative frequency detuning \([11, \omega, 170]\).

An improved first approximation, similar to \([11]\), can be obtained from the equilibrium condition

\[
S'_{x_n} = S'_{y_n} = 0 \quad \text{при} \quad \epsilon_r \cong 0.
\]

Substituting (3), (5) into (18), we obtain

\[
\begin{align*}
x_a & = \frac{1}{\pi n^2 \omega} \int_{0}^{2\pi} \frac{\partial U}{\partial x} \cos \left( \omega t \right) d\tau, \\
y_a & = \frac{1}{\pi n \omega} \int_{0}^{2\pi} \frac{\partial U}{\partial x} \sin \left( \omega t \right) d\tau,
\end{align*}
\]

where in a first approximation \( x \cong x_0 + x_1 \cos \left( \alpha \tau \right) - \left( y_1 / \alpha \right) \sin \left( \alpha \tau \right) \).
We return to equation (1), we will seek a solution in the form (13), using the representation

\[
\cos x = \text{Re} \left[ \exp[i\varphi(x)] \right],
\]
formulas (13) and [24]

\[
\exp \left[ i r_n \cos \left( \pi x - \varphi_n \right) \right] = \sum_{k=0}^{\infty} I_{2k} \left( r_n \right) \exp \left[ i k \left( \pi x - \frac{\pi}{2} - \varphi_n \right) \right].
\]

(21)

\[
S = \sum_{n=1}^{\infty} \frac{\mu_n^2 \tau_1}{4} - \frac{\mu_1^2}{4} + \frac{1}{2} \sum_{k_1, k_2, \ldots, k_m} \prod_{n=1}^{\infty} I_{k_n} \left( r_n \right) \sum_{\beta=1} \sum_{\gamma=1}^m \delta_{\beta\gamma} \sum_{n=1}^{\infty} k_n \times \\
\times (1 + \delta_{\beta\gamma}) \cos \left[ x_0 + \sum_{n=1}^{\infty} k_n \left( \tau_1 - \delta_{\beta\gamma} \tau_\varphi \right) - \delta_{\beta\gamma} \left( \pi \pm \varphi \right) \right],
\]

(22)

where \( J_k \left( r_n \right) \) — Bessel functions, \( \delta_{\beta\gamma} \) — Kronecker symbol.

Often, as experience shows, it is sufficient to limit oneself to the contribution to \( S \) (22) of several terms, in particular, of \( n = 1 \). This is quite sufficient for practical calculations without significant loss of accuracy [25], since series (22) quickly converges due to the well-known property of Bessel functions to rapidly decrease with increasing index for a fixed value of the argument \( r_n \).

In the general case \( U \left( x_1, \tau \right) \), the convergence of the series (5) will be determined by the boundedness of the functions under the integrals (19), (20).

The search for periodic solutions of equations of type (1), as follows from (6), (13), (16), reduces to finding and investigating the stability of critical points (22) with respect to \( r_n, \varphi_n \) or \( x_n, \tau_n \) in \( x_0, y_0 \).

We consider various cases of solutions of (1). In the simplest case of a mathematical pendulum without taking into account friction and vibrations, the results of calculations (13) for \( S \) (22) with \( n=1 \)

\[
S = \left[ \frac{\alpha_1^2 \tau_1}{4} - \frac{\mu_1^2}{2} + \varepsilon_0 J_0 \left( r_1 \right) \cos x_0 \right],
\]

(23)

quite satisfactory accuracy. The relative error of approximation \( a \left( r_1 \right) \) even at angles of deviation of the pendulum \( x = 160^\circ \) does not exceed 5.5\% [11, 0.55]

The introduction of longitudinal vibration, as follows

\[
S = \left[ \frac{\alpha_1^2 \tau_1}{4} - \frac{\mu_1^2}{2} + \varepsilon_0 J_0 \left( r_1 \right) \cos x_0 + \varepsilon_1 J_1 \left( r_1 \right) \cos \left( x_0 + \frac{\pi}{2\alpha} \right) \cos \frac{\varphi_1}{\alpha} \right],
\]

(24)

and (13), leads to the appearance of two types of critical points. The first correspond to the equilibrium positions \( x_0 = \pm n\pi, \varphi_1 = 0, \pm \pi/2, 1/\alpha \), the second - \( x_0 \neq \pm n\pi, \varphi_1 = 0, \pm \pi \) (1/\alpha - odd), \( n = 0, 1, 2, \ldots \) (in particular, \( x_0 = + (2n + 1) \) when \( n_0 = 0 \)).

Therefore, taking into account the scenario of “merging” of two periodic Poincaré solutions (II) due to the presence of the second type of critical points \( x_0 \neq \pm n\pi \) (bifurcation of the period \( 1/\alpha = 2 \leftrightarrow 1/\alpha = 1 \)), we will seek a solution to the problem of the
From the first bracket (36) we obtain an estimate of the upper boundary of the stable solution \( x_0 \approx x_0 + r_1 \cos (\tau/2 - \varphi_1) + r_2 \cos (\tau - \varphi_2). \) \( \tag{25} \)

Such a representation (16) gives the expression \( S \) (22) up to \( n=2 \)

\[
S \approx \left[ \frac{r_1}{16} + \frac{r_1}{4} - \frac{r_1}{2} \right] + \sum_{i=1}^{n} J_{2i}(r_i) J_{2i}(r_2) \cos \left( 2\varphi_1 + 2\varphi_2 \right) - \sum_{i=1}^{n} J_{2i}(r_i) J_{2i}(r_2) \cos \left( 2\varphi_1 + 2\varphi_2 \right)
+ \sum_{i=1}^{n} J_{2i}(r_1) J_{2i}(r_2) \cos (\varphi_1 - \varphi_2) \cos \left( 2\varphi_1 + 2\varphi_2 \right). \] \( \tag{26} \)

Restricting ourselves to terms of order \( r_4^4 \) in the expansion of \( J_{4}(r) \) and using the variables \( x_k, y_k \) (14), we obtain

\[
S \approx \left[ \frac{r_1}{16} + \frac{r_1}{4} - \frac{r_1}{2} \right] + \sum_{i=1}^{n} J_{2i}(r_i) J_{2i}(r_2) \cos \left( 2\varphi_1 + 2\varphi_2 \right) - \sum_{i=1}^{n} J_{2i}(r_i) J_{2i}(r_2) \cos \left( 2\varphi_1 + 2\varphi_2 \right)
+ \sum_{i=1}^{n} J_{2i}(r_1) J_{2i}(r_2) \cos (\varphi_1 - \varphi_2) \cos \left( 2\varphi_1 + 2\varphi_2 \right). \] \( \tag{27} \)

Substituting \( S \) (26) in (7), for \( S \approx 0 \), \( \sin x_0 = y_0 \), \( x_1 = y_1 \), \( x_2 = y_2 \), \( x_3 = y_3 \), \( y_4 = y_5 = 0 \), we obtain the corresponding equations for finding the equilibrium points and the characteristic roots \( \lambda_0 \)

\[
S_{x_1} \approx S_{y_1} \approx S_{y_2} \approx S_{y_3} \approx S_{y_4} \approx 0,
S_{x_2} \approx x_2 \left[ 1 - 4z_2 \left( 1 - \frac{x_2}{6} \right) - 2z_2 \left( 1 - \frac{x_2}{6} \right) \right] \approx 0.
S_{y_2} \approx y_2 \left[ 1 - 4z_2 \left( 1 - \frac{x_2}{6} \right) - 2z_2 \left( 1 - \frac{x_2}{6} \right) \right] \approx 0.
\( \lambda + S_{x_2} = 0, \quad \lambda + S_{y_2} = 0 \),
\( \lambda + S_{x_3} = 0, \quad \lambda + S_{y_3} = 0 \).
\( \lambda = \lambda_0 + \epsilon_0, \quad S_{x_1} = \epsilon_0 \approx \cos x_0 = S_{y_1} = f(x_1, y_1, \epsilon_0, i). \) \( \tag{31} \)

In the case \( x_1 = y_1 = 0 \), expressions (32), (33) are identically 0 and

\[
\lambda = 0 + \epsilon_0, \quad \lambda_0 = \epsilon_0 \cos x_0 = S_{y_1} = f(x_1, y_1, \epsilon_0, i). \] \( \tag{32} \)

From the first bracket (36) we obtain an estimate of the upper boundary of the stable solution \( \lambda = \epsilon_0 \approx (1 - 4z_2) \), from the second \( \epsilon_0 \approx 2 \), which is in agreement with the results obtained earlier by other methods for the Kapitsa pendulum \( \approx \) ouside the zone of parametric resonance \( \approx 2 \).

In the case \( x_1 = y_1 = 0 \), \( y_2 = 0 \) (\( x = 0, \ y_1 \neq 0 \)) from the conditions \( S_{x_1} = 0, S_{y_1} = 0 \) (34)–(35) we can be obtained
where \( f_{x, y}(\lambda) \) are expressions in square brackets (34).

It follows from (38) that there are two stable states of motion of the Kapitsa pendulum \( (\epsilon_0 < 0) \) in the zone of parametric resonance \( 2\epsilon_1^k \geq \frac{4}{\lambda} \mid \epsilon_0 \mid + 1, \quad (2 \mid \epsilon_1 \mid > \frac{4}{\lambda} \mid \epsilon_0 \mid + 1) \), differing from each other only by changing the sign of \( \epsilon_1^k \). The result with \( \gamma_1, \gamma_2 \neq 0 \), (37) \( \parallel \epsilon_0 \parallel = 0 \) was previously obtained by the Krylov – Bogolyubov method \( [1p, 281] \) without taking into account \( x_0, x_2, y_2, y_0 \) and the corresponding stability analysis.

This approach is not correct, since dropping the terms with \( x_2, y_2 \) in (25) at the frequency of the perturbing force leads, as follows from (34), (35). To the incorrect conclusion about the instability of the excited oscillations of the Kapitsa pendulum in the resonance zone with respect to \( x_0, y_0 \), which contradicts the experiment and the results of numerical simulation [12].

In the simplest case with \( \epsilon_0 = 0 \), the bifurcation point \( 4/\alpha = 2 \leftrightarrow 4/\alpha = 2 \) is found from a joint consideration of two periodic solutions according to scenario (II). Carrying out calculations similar to (31) - (38), near the equilibrium point \( x_0 = + (2 \ll 1) \pi / 2, \quad x_1 - y_1 = y_0 \wedge = 0 \), we obtain

\[
\begin{align*}
S_{x,x} \cdot S_{x,y} - S_{x,y} \cdot S_{y,x} &= 0, \\
S_{x,x} \cdot S_{y,y} &= \frac{x_0}{2} (1 - \frac{\pi}{2}), \\
S_{x,y} \cdot S_{y,y} &= \frac{1}{10} (1 + 2\pi x_0^2), \\
x_2 &= \frac{x_0}{2} (1 - (1 + \frac{\pi}{2} x_0^2)), \quad \epsilon_1^* = \pm 1 \sin x_0, \quad y_2 = 0.
\end{align*}
\]

Periodic solutions with \( \alpha_4 = \pm 1 \) \( \mid x_3 \mid < \pi/2 \) are unstable with respect to \( x_0, y_0 \mid \exp \mid \lambda \mid \cdot \cdot \cdot \), since \( S_{x,x} \cdot S_{y,y} < 0 \). Solving together (37), (42), one can determine the corresponding bifurcation point from the condition (see pic.)

\[
\begin{align*}
\mid x_i^* (\epsilon_0) \mid + \mid x_i^* (\epsilon_0^* \mid = \frac{\pi}{2}, \\
x_i^* = 50^\circ, \quad x_i^* = 31^\circ, \quad \epsilon_1^* = 0.61.
\end{align*}
\]

In this case \( \epsilon_0 = 0 \), the appearance of bifurcation can simultaneously lead to chaos in system (1) (see figure). The reason may be fluctuations, errors from the macro-system used in the physical, analog or numerical modeling of the deterministic system described by equation (1). As a result, cascades of transitions between different types of periodic motions at \( \gamma_1 = \epsilon_1^* \) (vibrational 1: 2, 1: 1; rotational 1: 1, etc.), which are perceived as chaos, will be observed.
Machine simulation of equation (1) at the ACWC GVS of the «Rusalka» hot water heater and full-scale simulation on a magnetic needle from a compass placed in a magnetic field confirmed the correctness of the results obtained within the limits of modeling errors.

At the end of his work [2] P.L. Kapitsa noted that the orienting moment’s arising from vibrational processes escaped the attention of physicists, so it would be interesting to raise the question of the possibility of observing the orienting effect of the vibrational moment on particles "and the molecules."

The scenario of the appearance of a bifurcation for an inverted pendulum according to Poincaré at $\varepsilon_0 = 0$, $0 - 0.6 < \varepsilon, \sigma - \text{dependences} x, y (\varepsilon_j)$.

Only later was such a possibility realized in a number of works on the confinement and capture of charged particles [3, 26, 27], particles with magnetic moment [28, 29] outside the zone and in the zone of parametric resopaps [25,30,31] in inhomogeneous electromagnetic fields.

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Institute of Physics and Technology
Ural Academy of Sciences of the USSR
Izhevsk

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Translated by Shironosova O. E.
Found a mistake?
Write me: shironosova.pr@gmail.com