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ABOUT PENDULUM P.L. KAPITSA OUTSIDE AND IN THE AREA OF PARAMETRIC RESONANCE

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Based on the resonant theory of dynamical Poincare systems, the motion is studied P.L. Kapitsa's inverted pendulum with a vibrating suspension point outside and in the zone of parametric resonance. General formulas for finding periodic motions and studying their stability without assuming smallness are obtained in an analytical form. Amplitudes of the oscillating pendulum.

Using the example of two types of solutions (stable (2: 1) and unstable (1: 1)), we obtained in the analytical form, the conditions for the occurrence of chaos and bifurcation points 2:1 <> 1:1 for inverted pendulum.

The importance of studying the dynamic stability of unstable states is noted nonlinear systems like a pendulum outside and in the zone of linear parametric resonance for holding and trapping atomic particles in electrodynamic traps.

The problem of the dynamics of a pendulum with a vibrating suspension point for a long time attracts attention [1-13]. This is due to the fact that the corresponding equation as a model

$$\ddot{x}+\varepsilon_r\dot{x}+(\varepsilon_0+\varepsilon_1\cos\tau)\sin x-\varepsilon_{-1}\cos(\tau+\varphi)\cos x=0, \qquad (1)$$

it is quite often found in various fields of physics: mechanics, electrodynamics, plasma physics, etc. In particular, for $\varepsilon_\beta=a_\beta/(\omega^2 l),\quad \beta= \ =0,\ \pm1,$, где a_0, where a_0 is the acceleration of gravity, $a_{1(-1)}$ is the amplitude of the longitudinal (transverse) vibration, *I* is the length of the pendulum, for a particle with an intrinsic magnetic moment $\mu,\ \varepsilon_\beta=\mu H_\beta/I,$ where T is the moment of inertia, H_0 is the intensity of the constant magnetic field, and $H_{1(-1)}$ is the amplitude of the variable of the longitudinal (transverse) pumping magnetic field, $\psi=\mathrm{const},\quad \tau=\omega t.$ For small deflection angles *x* and $e_{-1}=0$, equation (1) reduces to the well-known hurray to Mathieu's theory, which admits a stable state of an inverted pendulum $(\varepsilon_0<0,\ \varepsilon_1\neq0)$ outside the zone of parametric resonance. In 1950, P. L. Kapitsa [2], using the approximate solution method, described and experimentally demonstrated this effect. Based on numerical modeling, the authors of [12] found stable parametrically excited oscillations of an inverted pendulum in the resonance zone. Later, [1, 7], the corresponding dependences of the oscillation amplitudes on s_0, e_1 were obtained.

In addition to the above, many other non-trivial solutions were considered: vibrational, vibrational-rotational [1,2, 7, 11 , 12]; the emergence of chaos [8,10], etc. The search for solutions (1), as a rule, for various cases was carried out using various methods (Cesari [4, 6], Krylov-Bogolyubov [11], through action-angle variables [8], etc. [14, 15] with the expansion of *sin x*, *cos x* in a series in powers of smallness *x*. Such a variety of methods made it difficult to stitch together particular solutions, interpret the obtained results, and understand the causes of chaos and bifurcations in systems described by equations of type (1).

Therefore, given the two provisions of Poincare [13, 75] that ". . . periodic solutions are the only breach through which we could try to penetrate into an area that was considered inaccessible "(I) and that". . . The periodic solution can disappear only by merging with another periodic solution ", that is,". . . Periodic solutions disappear in pairs like the real roots of algebraic equations "(II), we use a generalization of the corresponding methods to find and study the stability of periodic solutions (1) from critical points of the action function [13, 16-23].

To do this, we rewrite equation (1) in Lagrangian form

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=-\frac{\partial F}{\partial \dot{x}}, \tag{2}$$

$$L=T-U,\ T=\dot{x}^2/2,\ F=\varepsilon_r\dot{x}^2/2, \tag{3}$$

$$U=-(\varepsilon_0+\varepsilon_1\cos\tau)\cos x-\varepsilon_{-1}\cos(\tau+\varphi)\sin x. \tag{4}$$

In the general case, x can be a vector and $U=U(\bar{x},\tau)$.We will seek a solution to (2) near a periodic solution at a frequency in the form of *a* series

$$x=x_0+\sum_{n=1}^{\infty}\left[x_n\cos(n\alpha\tau)+\frac{y_n}{n\alpha}\sin(n\alpha\tau)\right], \tag{5}$$

where x_0, x_n, y_n in the general case *f (x)*.

Given the dependence x, x = f (x_k, y_k, x_k, y_k), we can obtain the following shortened equations in the approximation of slowly varying amplitudes x_k, y_k for the period $2\pi/a$:

$$\dot{x}_k\simeq-\frac{\partial S}{\partial y_k}-\frac{\partial R}{\partial x_k},\quad \dot{y}_k\simeq\frac{\partial S}{\partial x_k}-\frac{\partial R}{\partial y_k}, \tag{6}$$

где $y_0=\dot{x}_0$, $k=1,2,\dots,\infty$ и

$$S=s-y_0^2,\ s=\langle L\rangle=\frac{\alpha}{2\pi}\int_0^{2\pi/\alpha}L\,d\tau, \tag{7}$$

$$R=\frac{\varepsilon_r}{2}\left[y_0^2+\frac{1}{2}\sum_{n=1}^{\infty}(x_n^2+y_n^2)\right]. \tag{8}$$

When deriving (6), the formulas were taken into account

$$\left\langle \frac{\partial L}{\partial x_n} \right\rangle \cong \left\langle \left[\frac{\partial L}{\partial x} \cos(n\alpha\tau) + \frac{\partial L}{\partial \dot{x}} \frac{d}{d\tau} \cos(n\alpha\tau) \right] \right\rangle, \tag{9}$$

$$\left\langle \frac{\partial L}{\partial y_n} \right\rangle \cong \left\langle \left[\frac{1}{n\alpha} \frac{\partial L}{\partial x} \sin(n\alpha\tau) + \frac{\partial L}{\partial \dot{x}} \frac{d}{d\tau} \sin(n\alpha\tau) \right] \right\rangle, \tag{10}$$

$$\left\langle \frac{\partial L}{\partial x_0} \right\rangle \cong \left\langle \frac{\partial L}{\partial x} \right\rangle, \quad \left\langle \frac{\partial L}{\partial y_0} \right\rangle \cong \left\langle \frac{\partial L}{\partial \dot{x}} \right\rangle, \tag{11}$$

$$\ddot{x} \cong \ddot{x}_0 - n\alpha\,(2\dot{x}_n + y_n) \sin(n\alpha\tau) + (2\dot{y}_n - n^2\alpha^2 x_n) \cos(n\alpha\tau) \tag{12}$$

and conditions for the extremality of the action function (2). In the amplitude – phase variables, equations (6) take the form

$$\dot{\psi}_n \cong \frac{1}{n r_n} \frac{\partial S}{\partial r_n}, \quad \dot{r}_n \cong -\frac{1}{n r_n} \frac{\partial S}{\partial \psi_n} - \varepsilon_r r_n, \tag{13}$$

$$x_n = r_n \cos \psi_n, \quad y_n/n\alpha = r_n \sin \psi_n, \tag{14}$$

$$x = x_0 + \sum_{n=1}^{\infty} r_n \cos(n\alpha\tau - \psi_n). \tag{15}$$

In action-angle variables

$$\dot{\psi}_n \cong \frac{\partial S}{\partial \chi_n}, \quad \dot{\chi}_n \cong -\frac{\partial S}{\partial \psi_n} - 2\varepsilon_r \chi_n, \tag{16}$$

$$x_n = x_0 + \sum_{n=1}^{\infty} \left(\frac{2\chi_n}{n} \right)^{1/2} \cdot \cos(\alpha\tau n - \psi_n). \tag{17}$$

It is easy to show that, to a first approximation, the Krylov – Bogolyubov method I[11,14] and the S-function method for $\eta = 1$ lead to the same shortened equations for r_1 and φ_1. To do this, it is sufficient to substitute (6) into (15) and take into account the equalities $\langle \partial U/\partial r_1 \rangle \cong \langle \partial U/\partial x \cos(\alpha\tau - \psi_1) \rangle$, $\langle \partial U/\partial \psi_1 \rangle \cong \langle \partial U/\partial x \sin(\alpha\tau - \psi_1) \rangle$. The parameter of smallness in both cases will be the relative frequency detuning [11, °, 170].

An improved first approximation, similar to [11], can be obtained from the equilibrium condition $S'_{x_n} = S'_{y_n} = 0$ при $\varepsilon_r \cong 0$

$$\frac{\partial S}{\partial x_n,\, y_n} = \frac{\partial \langle T \rangle}{\partial x_n,\, y_n} - \frac{\partial \langle U \rangle}{\partial x_n,\, y_n} = 0. \tag{18}$$

Substituting (3), (5) into (18), we obtain

$$x_n = \frac{1}{\pi n^2 \alpha} \int_0^{2\pi/\alpha} \frac{\partial U}{\partial x} \cos(n\alpha\tau)\, d\tau, \tag{19}$$

$$y_n = \frac{1}{\pi\alpha} \int_0^{2\pi/\alpha} \frac{\partial U}{\partial x} \sin(n\alpha\tau)\, d\tau, \tag{20}$$

where in a first approximation $x \cong x_0 + x_1 \cos(\alpha\tau) + (y_1/\alpha) \sin(\alpha\tau)$.

We return to equation (1), we will seek a solution in the form (13), using the representation $\cos x = \mathrm{Re}\,[\exp(ix)]$, formulas (13) и [24]

$$\exp[ir_n \cos(n\alpha\tau - \psi_n)] = \sum_{k_n=-\infty}^{+\infty} J_{k_n}(r_n) \exp[ik_n(n\alpha\tau + \pi/2 - \psi_n)], \tag{21}$$

им

$$S = \sum_{n=1}^{\infty} \frac{n^2\alpha^2 r_n^2}{4} - \frac{y_0^2}{2} + \frac{1}{2} \sum_{k_1, k_2 \ldots = -\infty}^{+\infty} \cdot \prod_{n=1}^{+\infty} J_{k_n}(r_n) \cdot \sum_{\beta=-1}^{+1} \varepsilon_\beta \cdot \delta^{\mp\beta}_{\sum_{n=1}^{\infty} k_n n\alpha} \times$$
$$\times (1 + \delta_\beta^0) \cdot \cos\left[x_0 + \sum_{n=1}^{\infty} k_n\left(\frac{\pi}{2} - \delta_\beta^{\pm 1}\psi_n\right) - \delta_\beta^{-1}(\pi/2 \pm \varphi)\right], \tag{22}$$

where $J_k(r_n)$ — Bessel functions, δ_β^n — Kronecker symbol.

Often, as experience shows, it is sufficient to limit oneself to the contribution to S (22) of several terms, in particular, of n = 1. This is quite sufficient for practical calculations without significant loss of accuracy [25], since series (22) quickly converges due to the well-known property of Bessel functions to rapidly decrease with increasing index for a fixed value of the argument r_n.

In the general case $U(x, \tau)$, the convergence of the series (5) will be determined by the boundedness of the functions under the integrals (19), (20).

The search for periodic solutions of equations of type (1), as follows from (6), (13), (16), reduces to $\varepsilon_r \cong 0$ finding and investigating the stability of critical points (22) with respect to r_n, ψ_n or $\varkappa_n$, $\Psi_n(x_n, y_n)$ и x_0, y_0.

We consider various cases of solutions of (1). In the simplest case of a mathematical pendulum without taking into account friction and vibrations, the results of calculations (13) for S (22) with n=1

$$S \cong \left[\frac{\alpha^2 r_1^2}{4} - \frac{y_0^2}{2} + \varepsilon_0 J_0(r_1) \cos x_0\right], \tag{23}$$

quite satisfactory accuracy. The relative error of approximation a (r_1) even at angles of deviation of the pendulum x =160 ° does not exceed 5.5% [11,0,55]

The introduction of longitudinal vibration, as follows

$$S \cong \left[\frac{\alpha^2 r_1^2}{4} - \frac{y_0^2}{2} + \varepsilon_0 J_0(r_1) \cos x_0 + \varepsilon_1 J_{1/\alpha}(r_1) \cos\left(x_0 + \frac{\pi}{2\alpha}\right) \cos\frac{\psi_1}{\alpha}\right] \tag{24}$$

and (13), leads to the appearance of two types of critical points. The first correspond to the equilibrium positions $x_0 = \pm n\pi,\ \psi_1 = 0,\ \pm\pi/2,\ 1/\alpha$, the second - $x_0 \neq \pm n\pi, \psi_1 = 0, \pm\pi\ (1/\alpha$ - odd) , n = 0, 1,2,. . . (in particular, x_0 = +- (2n + 1) when n_o = 0).

Therefore, taking into account the scenario of “merging” of two periodic Poincaré solutions (II) due to the presence of the second type of critical points $x_0 \neq \pm n\pi$ (bifurcation of the period $1/\alpha = 2 \leftrightarrow 1/\alpha = 1$), we will seek a solution to the problem of the

Kapitsa pendulum outside and in the zone of parametric resonance in the form

$$x_0 \cong x_0 + r_1 \cos(\tau/2 - \psi_1) + r_2 \cos(\tau - \psi_2). \tag{25}$$

Such a representation (16) gives the expression S (22) up to n=2

$$S \cong \left\{ \frac{r_1^2}{16} + \frac{r_2^2}{4} - \frac{y_0^2}{2} + \varepsilon_0 \left[J_0(r_1) J_0(r_2) \cos x_0 + \right.\right.$$
$$\left. + 2 \sum_{n=1}^{\infty} J_{2n}(r_1) J_n(r_2) \cos\left(x_0 - \frac{n\pi}{2}\right) \cos n(2\psi_1 - \psi_2) \right] - \varepsilon_1 \left[J_2(r_1) J_0(r_2) \cos 2\psi_1 + \right.$$
$$\left.\left. + \sum_{n=1}^{\infty} J_{2n\pm2}(r_1) J_n(r_2) \cos(x_0 - n\pi/2) \cos[n(2\psi_1 - \psi_2) \pm 2\psi_1] \right] \right\}. \tag{26}$$

Restricting ourselves to terms of order r_k^4 in the expansion of $J_n(r_k)$ в S (26) and using the variables x_k, y_k (14), we obtain

$$S \cong \left[\frac{x_1^2}{16} + \frac{y_1^2}{4} + \frac{x_2^2}{4} + \frac{y_2^2}{4} - \frac{y_0^2}{2} + (\varepsilon_0 f_0 - \varepsilon_1 f_1) \cos x_0 + (\varepsilon_0 F_0 - \varepsilon_1 F_1) \sin x_0 \right], \tag{27}$$

где

$$f_0 = \left[1 - \frac{x_1^2 + 4y_1^2 + x_2^2 + y_2^2}{4} + \frac{(x_1^2 + 4y_1^2)^2 + (x_2^2 + y_2^2)^2}{64} + \frac{(x_1^2 + 4y_1^2)(x_2^2 + y_2^2)}{16} \right], \tag{28}$$

$$f_1 = \left[(x_1^2 - 4y_1^2) \frac{8 - \frac{2}{3}(x_1^2 + 4y_1^2) - (x_2^2 + 3y_2^2)}{64} - \frac{x_1 y_1 x_2 y_2}{8} \right], \tag{29}$$

$$F_0 = \frac{4x_1 y_1 y_2 + x_2(x_1^2 - 4y_1^2)}{8}, \quad F_1 = x_2 \left[\frac{1}{2} - \frac{x_1^2 + 4y_1^2}{8} - \frac{x_1^2 + y_2^2}{16} \right]. \tag{30}$$

Substituting S (26) in (7), for $\varepsilon_r \cong 0,\ \sin x_0 = x_2 = y_2 = x_1 y_1 = y_0 = 0$ in we obtain the corresponding equations for finding the equilibrium points and the characteristic roots λ_0

$$S'_{x_2} \cong S'_{y_2} \cong S'_{y_0} \cong S'_{x_0} \cong 0, \tag{31}$$

$$S'_{x_1} = x_1 \left[1 - 4\varepsilon_0^{\pm} \left(1 - \frac{x_1^2}{8} - \frac{y_1^2}{2} \right) - 2\varepsilon_1^{\pm} \left(1 - \frac{x_1^2}{6} \right) \right] \cong 0. \tag{32}$$

$$S'_{y_1} = y_1 \left[1 - 4\varepsilon_0^{\pm} \left(1 - \frac{x_1^2}{8} - \frac{y_1^2}{2} \right) + 2\varepsilon_1^{\pm} \left(1 - \frac{2}{3} y_1^2 \right) \right] \cong 0, \tag{33}$$

$$(\lambda^2 + S''_{x_1x_1}) [(\lambda^2 + S''_{x_2x_2} S''_{y_2y_2}) \cdot (\lambda^2 + S''_{x_0x_0} S''_{y_0y_0}) - S''_{y_2y_2} S''_{y_0y_0} S''^{2}_{x_0x_2}], \tag{34}$$

$$\lambda = \lambda_0 + \varepsilon_r, \quad \varepsilon_{0,1}^{\pm} = \varepsilon_{0,1} \cos x_0 \ \text{и}\ S''_{i,j} = f(x_1, y_1, \varepsilon_{0,1}^{\pm}). \tag{35}$$

In the case x_1=y_1 = 0 expressions (32), (33) are identically 0 and

$$\left\{ \lambda^2 + \frac{1}{16} [(1 - 4\varepsilon_0^{\pm})^2 - 4\varepsilon_1^{\pm 2}] \right\} \left\{ \lambda^4 + \frac{\lambda^2}{4} (1 + \varepsilon_0^{\pm})^2 + \right.$$
$$\left. + \frac{1}{8} (1 - \varepsilon_0^{\pm}) [\varepsilon_1^{\pm 2} + 2\varepsilon_0^{\pm} (1 - \varepsilon_0^{\pm})] \right\}. \tag{36}$$

From the first bracket (36) we obtain an estimate of the upper boundary of the stable solution $4(\varepsilon_1^{\pm})^2 < (1 - 4\varepsilon_0^{\pm})^2$, from the second - $(\varepsilon_1^{\pm})^2 > 2 \mid \varepsilon_0^{\pm} (1 - \varepsilon_0^{\pm})$ |, which is in agreement with the results obtained earlier by other methods for the Kapitsa pendulum $(\varepsilon_1^{\pm} < 0)$ outside the zone of parametric resonance [2, 15].

In the case $x_1 \neq 0,\ y_1 = 0\ (x = 0,\ y_1 \neq 0)$ from the conditions $S'_{x_1} = 0\ (S'_{y_1} = 0)\ (31)-(35)$ we can be obtained

$$x_1^2 = 6\left[\frac{4\varepsilon_0^\pm + 2\varepsilon_1^\pm - 1}{2\varepsilon_1^\pm + 3\varepsilon_0^\pm}\right], \quad \left(y_1^2 = \frac{4\varepsilon_0^\pm - 2\varepsilon_1^\pm - 1}{3\varepsilon_0^\pm - 2\varepsilon_1^\pm}\frac{3}{2}\right), \tag{37}$$

$$\left[\lambda^2 + \frac{x_1^2}{24}(2\varepsilon_0^\pm + \varepsilon_1^\pm + 1)\right] f_x(\lambda) = 0, \quad \left[\left[\lambda^2 - \frac{\varepsilon_1^\pm}{6} y_1^2 (2\varepsilon_0^\pm - 2\varepsilon_1^\pm - 1)\right] f_y(\lambda) = 0\right], \tag{38}$$

where $f_{x,y}(\lambda)$ are expressions in square brackets (34).

It follows from (38) that there are two stable states of motion of the Kapitsa pendulum $(\varepsilon_0^\pm < 0)$ in the zone of parametric resonance $2\varepsilon_1^\pm > 4\,|\,\varepsilon_0^\pm\,|+1$, $(2\,|\,\varepsilon_1^\pm\,| > 4\,|\,\varepsilon_0^\pm\,|+1)$, differing from each other only by changing the sign of $\varepsilon_1^\pm$. The result with $y_1 \neq 0$, (37) и $\varepsilon_0 = 0$ was previously obtained by the Krylov – Bogolyubov method [1p,281] without taking into account x_0, x_2, y_2, y_0 and the corresponding stability analysis.

This approach is not correct, since dropping the terms with x_2, y_2 in (25) at the frequency of the perturbing force leads, as follows from (34), (35). To the incorrect conclusion about the instability of the excited oscillations of the Kapitsa pendulum in the resonance zone with respect to x_0 , y_0, which contradicts the experiment and the results of numerical simulation [12].

In the simplest case with $\varepsilon_0 = 0$, the bifurcation point $1/\alpha = 2 \leftrightarrow 1/\alpha = 1$ is found from a joint consideration of two periodic solutions according to scenario (II). Carrying out calculations similar to (31) - (38), near the equilibrium point x0 = + (2 «-) - 1) π / 2, x1 - y1 = y0 ^ = 0 $x_0 = \pm(2n+1)\,\pi/2$, $x_1 = y_1 = y_0 = 0$, we obtain

$$\prod_{k=0}^{2}\left(\lambda^2 + S''_{x_k x_k} S''_{y_k y_k} - \delta_k^2 S''^2_{x_k y_k}\right) = 0, \tag{39}$$

$$S''_{x_0 x_0} \cdot S''_{y_0 y_0} = -\frac{\varepsilon_1^* x_2}{2}\left(1 - \frac{x_2^2}{8}\right), \quad S''_{x_1 x_1} \cdot S''_{y_1 y_1} = \frac{1}{16}(1 + 2\varepsilon_1^* x_2)^2, \tag{40}$$

$$S''_{x_2 x_2} \cdot S''_{y_2 y_2} - S''^2_{x_2 y_2} = \frac{1}{4}\left(1 + \frac{3\varepsilon_1^* x_2}{4}\right), \tag{41}$$

$$x_2 \cong \frac{4}{3\varepsilon_1^*}\left(1 - \left(1 + \frac{3}{2}\varepsilon_1^{*2}\right)^{1/2}\right), \quad \varepsilon_1^* = \varepsilon_1 \sin x_0, \quad y_2 = 0. \tag{42}$$

Periodic solutions with $\alpha^{-1} = 1$ при $|\,x_2\,| < \pi/2$ are unstable with respect to x_0, y_0 $[\exp|\,\lambda\,|\,\tau]$, since $S''_{x_0 x_0} \cdot S''_{y_0 y_0} < 0$. Solving together (37), (42), one can determine the corresponding bifurcation point from the condition (see pic.)

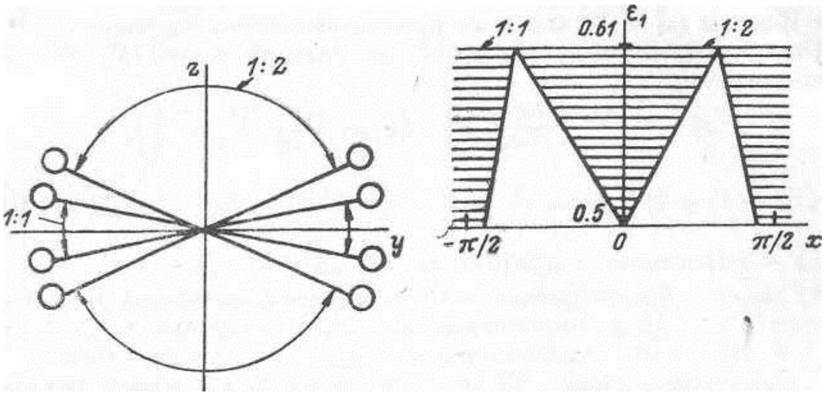
$$|\,x_1^*(\varepsilon_1')\,| + |\,x_2^*(\varepsilon_1')\,| = \frac{\pi}{2}, \tag{43}$$

$x_1^* \simeq 59°$, $x_2^* \simeq 31°$, $\varepsilon_1' \simeq 0.61$.

In this case $(с\ \varepsilon_0 = 0)$, the appearance of bifurcation can simultaneously lead to chaos in system (1) (see figure). The reason may be fluctuations, errors from the macro-system used in the physical, analog or numerical modeling of the deterministic system described by equation (1). As a result, cascades of transitions between different types of periodic motions at $\varepsilon_1 = \varepsilon_1'$ (vibrational 1: 2, 1: 1; rotational 1: 1, etc.), which are perceived as chaos, will be observed.

Machine simulation of equation (1) at the ACWC GVS of the «Rusalka» hot water heater and full-scale simulation on a magnetic needle from a compass placed in a magnetic field confirmed the correctness of the results obtained within the limits of modeling errors.

At the end of his work [2] P.L. Kapitsa noted that the orienting moment’s arising from vibrational processes escaped the attention of physicists, so it would be interesting to raise the question of the possibility of observing the orienting effect of the vibrational moment on particles "and the molecules.



The scenario of the appearance of a bifurcation for an inverted pendulum according to Poincare at $\varepsilon_0 = 0$. a — $0.5 < \varepsilon_1 < 0.61$, б — dependences $x_{1,2}(\varepsilon_1)$.

Only later was such a possibility realized in a number of works on the confinement and capture of charged particles [3, 26, 27], particles with magnetic moment [28, 29] outside the zone and in the zone of parametric resopaps [25,30,31] in inhomogeneous electromagnetic fields.

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